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Closed and σ-Finite Measures on the Orthogonal Projections

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We characterize the connection between closed and σ-finite measures on orthogonal projections of von Neumann algebras.

Let $\mathcal{A}$ be a von Neumann algebra acting in a separable complex Hilbert space H and let $\mathcal{A}^{\mathrm{pr}}$ be the set of all orthogonal projections (=idempotents) from $\mathcal{A}$. A subset $\mathcal{M} \subseteq \mathcal{A}^{\mathrm{pr}}$ is said to be *ideal of projections* if:

a) $p \leq q$ where $p \in \mathcal{A}^{\mathrm{pr}}, q \in \mathcal{M} \Rightarrow p \in \mathcal{M}$;
b) $p, q \in \mathcal{M}$ and $\|pq\| < 1 \Rightarrow p \vee q \in \mathcal{M}$; c) $\sup\{p : p \in \mathcal{M}\} = I$.

Put $\mathcal{M}_p := \{q : q \in \mathcal{M}, q \leq p\}$, $\forall p \in \mathcal{A}^{\mathrm{pr}}$. Note that $\mathcal{A}^{\mathrm{pr}}$ is the ideal of projections, $0 \in \mathcal{M}_p, \forall p$, and the conditions 1), 2) are fulfilled on $\mathcal{M}_p$.

A function $\mu : \mathcal{M} \to [0, +\infty]$ is said to be a measure if $\mu(e) = \sum \mu(e_i)$ for any representation $e = \sum e_i$. Let $\mu_1 : \mathcal{M}_1 \to [0, +\infty]$ and $\mu_2 : \mathcal{M}_2 \to [0, +\infty]$ be measures. The measure μ_2 is said to be the *continuation* of μ_1 if $\mathcal{M}_1 \subset \mathcal{M}_2$ and $\mu_1(p) = \mu_2(p), \forall p \in \mathcal{M}_1$. A projection $p \in \mathcal{A}^{\mathrm{pr}}$ is said to be: *projection of finite μ-measure* if $\sup\{q \in \mathcal{M}_p\} = p$ and $\sup\{\mu(q) : q \in \mathcal{M}_p\} < +\infty$; *hereditary projection of finite μ-measure* if q is the projection of finite μ- measure for any $q \in \mathcal{A}^{\mathrm{pr}}, q \leq p$.

The measure μ is said to be: *finite* if $\mu(p) < \infty, \forall p$; *infinite* if there exists $p \in \mathcal{M}$ such that $\mu(p) = +\infty$; *fully finite* if $\sup\{\mu(p) : p \in \mathcal{M}\} < +\infty$; *closed* if μ is finite and $p \in \mathcal{M}$ if p is the hereditary projection of finite μ-measure; *σ-finite* if $\mathcal{M} = \mathcal{A}^{\mathrm{pr}}$ and there exists a sequence $\{p_n\} \subset \mathcal{A}^{\mathrm{pr}}$ such that $p_n \nearrow I$ and $\mu(p_n) < +\infty, \forall n$.

The following Proposition will be needed in Theorem 3.

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Proposition 1. *Let $\mathcal{A}$ be a finite von Neumann algebra acting in the separable Hilbert space H and let $\mathcal{M} \subseteq \mathcal{A}^{\text{pr}}$ be the ideal of projections. Then there exists a sequence $\{e_n\} \subset \mathcal{M}$ such that $e_n \nearrow I$.*

Proof: Let τ be a faithful normal finite trace on $\mathcal{A}^+$. The proof consists of several steps.

i) Fix $\epsilon > 0$. Let us prove that there exists $p_\epsilon \in \mathcal{M}$ such that $\tau(p_\epsilon) > \tau(I) - \epsilon$. By b) and by separability of H, there exists a sequence $\{q_n\}_1^\infty \subset \mathcal{M}$ with $\sup\{q_n\} = I$.

1) Put $f_1 := q_1$.

2) Let $q_2 f_1 q_2 = \int_0^{1+} \lambda \mathrm{d}e_\lambda^{(2)}$ be the spectral decomposition of $q_2 f_1 q_2$. Here $e_\lambda^{(2)}$ is the left continuous decomposition of identity. Fix $k \in N$. By the left continuous of $e_\lambda^{(2)}$, there exists $\beta \in (0, 1)$ such that $\tau(e_1^{(2)} - e_\beta^{(2)}) < (\frac{1}{2})^{k+1}$. Put $q_2' := q_2 \wedge e_\beta^{(2)}$. By 1), $q_2' \in \mathcal{M}$. By the construction of β, $\|q_2' f_1\| \leq \beta$ (< 1). Hence $f_1 \vee q_2' \leq q_1 \vee q_2$ and by b), $f_2 := f_1 \vee q_2' \in \mathcal{M}$ and $\tau(q_1 \vee q_2 - f_2) < (\frac{1}{2})^{k+1}$.

3) Let $q_3 f_2 q_3 = \int_0^{1+} \lambda \mathrm{d}e_\lambda^{(3)}$ be the spectral decomposition of $q_3 f_2 q_3$. Let us choose $\beta \in (0, 1)$ such that $\tau(e_1^{(3)} - e_\beta^{(3)}) < (\frac{1}{2})^{k+2}$, again. Put $q_3' := q_3 \wedge e_\beta^{(3)}$. By a), $q_3' \in \mathcal{M}$. By the construction of β, we have $\|q_3' f_2\| \leq \beta$ (< 1). Again by b), $f_3 := f_2 \vee q_3' \in \mathcal{M}$. Thus

$$f_3 \leq f_2 \vee q_3 \leq f_1 \vee q_2 \vee q_3 = q_1 \vee q_2 \vee q_3$$

and

$$\tau(f_2 \vee q_3 - f_2 \vee q_3') \leq \left(\frac{1}{2}\right)^{k+2},$$

$$\times\, \tau(q_1 \vee g_2 \vee q_3 - f_2 \vee g_3) < \left(\frac{1}{2}\right)^{k+1}.$$

Therefore $\tau(q_1 \vee q_2 \vee q_3 - f_3) \leq (\frac{1}{2})^{k+1} + (\frac{1}{2})^{k+2}$.

Let us continue the process of construction of $\{f_n\}$ by the induction with respect to n.

n). Let the projection $f_{n-1} \in \mathcal{M}$ it was chosen. Let $q_n f_{n-1} q_n = \int_0^{1+} \lambda \mathrm{d}e_\lambda^{(n)}$ be the spectral decomposition of $q_n f_{n-1} q_n$. Let us choose $\beta \in (0, 1)$ such that $\tau(e_1^{(n)} - e_\beta^{(n)}) < (\frac{1}{2})^{k+n-1}$. Put $q_n' := q_n \wedge e_\beta^{(n)}$. By a), $q_n' \in \mathcal{M}$. By the construction of β, we have $\|q_n' f_{n-1}\| \leq \beta$ (< 1). By b), $f_n := f_{n-1} \vee q_n' \in \mathcal{M}$. Thus

$$f_n = f_{n-1} \vee q_n' \leq \vee_1^{n-1} q_i \vee q_n = \vee_1^n q_i$$

and

$$\tau(\vee_1^n q_i - f_n) \le \left(\frac{1}{2}\right)^{k+1} + \cdots + \left(\frac{1}{2}\right)^{k+n-1} < \left(\frac{1}{2}\right)^k.$$

For the given $\epsilon > 0$ let us choose $m \in N$ such that $\tau(I - \vee_1^m q_i) < \frac{\epsilon}{2}$ and $k \in N$ such that $\frac{\epsilon}{2} > (\frac{1}{2})^k$ $(> \tau(\vee_1^m g_i - f_m))$. Then the projection $p_\epsilon := f_m$ is that in question.

ii) Now let $e_n := \wedge_{m \ge n} p_{2^{-m}}$. Then $e_n^\perp = \vee_{m\ge n} p_{2^{-m}}^\perp$ and $\tau(e_n^\perp) \le \sum_{m\ge n} 2^{-m} = 2^{-n+1}$. The sequence $\{e_n\}$ is valid. □

Theorem 2. *Let $\mathcal{A}$ be a semifinite von Neumann algebra containing no direct summand of type I_2 acting in the separable Hilbert space and let $\mu : \mathcal{A}^{\text{pr}} \to [0, +\infty]$ be the σ-finite infinite measure and $\mathcal{M}_\mu := \{p \in \mathcal{A}^{\text{pr}} : \mu(p) < +\infty\}$. Then $\mathcal{M}_\mu$ is the ideal of projections. If $\mathcal{A}$ is a finite von Neumann algebra then the restriction $\mu_1 := \mu/\mathcal{M}_\mu$ is the closed measure.*

Proof:

1) Let us prove that $\mathcal{M}_\mu$ is the ideal of projections. It is clear that a) on $\mathcal{M}_\mu$ is fulfilled. Let $p, q \in \mathcal{M}_\mu$ and $\|pq\| < 1$. It is sufficient to consider the case p, q when p, q are projections *in general position* in H, i.e.

$$p \wedge q = (p \vee q - p) \wedge q = (p \vee q - q) \wedge p = 0. \qquad (1)$$

By (1), $\overline{pqH} = pH$.

i) Let us suppose first that projections p, q are finite with respect to $\mathcal{A}$. There exists a representation $q = q_1 + q_2 + q_3$ (if $\mathcal{A}$ is the continuous algebra then $q_3 = 0$) $q_1, q_2, q_3 \in \mathcal{M}$ such that the orthogonal projections p_i onto subspaces $\overline{p}q_i H$, $i = 1, 2, 3$ are mutually orthogonal and there exist the partial isometries $v_i \in \mathcal{A}$, $i = 1$, 2, 3 such that $q_i H$ are the initial subspaces and the final subspaces in $(q - q_i)H$. The von Neumann algebra $\mathcal{A}^i$ generated by p_i, q_i and v_i is direct integral of factors of type I_3. By the construction, $p_i, q_i, v_i q_i v_i^* \in \mathcal{M}$. By Lemma (Lugovaja and Sherstnev, 1980), $\mu(p_i \vee q_i - p_i) < +\infty$ (and hence $\mu(p_i \vee q_i) < +\infty$) if $\mathcal{A}^i$ is the type (Lugovaja and Sherstnev, 1980) factor. If $\mathcal{A}^i$ is the direct integral of factors, the proof of $\mu(p_i \vee q_i) < +\infty$ repeat of the proof of Lemma (Lugovaja and Sherstnev, 1980). Thus the inequality $\mu(p \vee q) = \mu(\sum_i p_i \vee q_i) < +\infty$ is proved. By Lemma 5 (Matvejchuk, 1981a),

$$\mu(p \vee q) \le (1 - \|pq\|)^{-1}(\mu(p) + \mu(q)). \qquad (2)$$

ii) Let us consider now the general case of $p, q \in \mathcal{M}$. Let $p_n \in \mathcal{M}$ be a sequence of finite projections, $p_n \nearrow p$ and let q_n be the orthogonal projection onto $\overline{qp_nH}$. The projection q_n is finite and $p_n \vee q_n \nearrow p \vee q$. The projections p_n and q_n are in the general position on the space $p_n \vee q_n H$. By (2),

$$\mu(p \vee q) = \lim \mu(p_n \vee q_n) \leq \lim(1 - \|p_n \vee q_n\|)^{-1}(\mu(p_n) + \mu(q_n)) \leq (1 - \|pq\|)^{-1})(\mu(p) + \mu(q)) < +\infty.$$

Hence $p \vee q \in \mathcal{M}_\mu$ and thus $\mathcal{M}_\mu$ is the ideal of projections.

2) Let $\{e_n\}$ be the sequence from Proposition 1. Then $p \wedge e_n \nearrow p, \forall p \in \mathcal{A}^{\text{pr}}$. If $\sup\{\mu_1(p \wedge e_n) : n\} < +\infty$ then $p \in \mathcal{M}_\mu$. Thus the set $\mathcal{M}_\mu$ contain any hereditary projection of finite μ-measure. By the definition, μ_1 is the closed measure. □

Theorem 3. *Let $\mathcal{A}$ be a finite von Neumann algebra containing no direct summand of type I_2 acting in the separable Hilbert space. Then any closed measure $\mu : \mathcal{M} \to [0, +\infty]$ can be extended to a σ-finite measure.*

Proof: Let $\mu : \mathcal{M} \to R$ be a closed measure. By Theorem (Matvejchuk, 1981b), any full finite measure can be extended by the strong operator topology to a unique fully finite measure on $\mathcal{A}^{\text{pr}}$. Now we may assume that the measure μ is not fully finite. Put $\mu_1(p) := +\infty$ for any $p \in \mathcal{A}^{\text{pr}} \backslash \mathcal{M}$ and $\mu_1(p) := \mu(p)$, for any $p \in \mathcal{M}$. Let us prove that the function $\mu_1 : \mathcal{A}^{\text{pr}} \to [0, +\infty]$ is a σ-finite measure. Let $p = \sum p_i$ be a decomposition of $p \in \mathcal{A}^{\text{pr}}$. By a) and by the definition of the measure, we have $\mu_1(p) = \sum_i \mu_1(p_i)$ for any $p \in \mathcal{M}$. Now let $p \in \mathcal{A}^{\text{pr}} \backslash \mathcal{M}$. Let us assume for the moment $\sum_i \mu_1(p_i) < +\infty$. By the finiteness of $\mathcal{A}$, the assumption gives us that p is the projections of finite μ measure. By Proposition 1, p is a hereditary projection of finite μ-measure. By the condition of the Theorem, $p \in \mathcal{M}$. We have the contradiction with $p \in \mathcal{A}^{\text{pr}} \backslash \mathcal{M}$. Therefore $\sum_i \mu_1(p_i) = +\infty = \mu_1(p)$. Let $\{e_n\}$ be the sequence from Proposition 1. Hence μ_1 is a σ-finite measure. □

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REFERENCES

Lugovaja, G. D. and Sherstnev, A. N. (1980). On the Gleasons theorem for unbounded measures. *Izvestija VUZov. Matematika*. **12**, 30–32. [in Russian].

Matvejchuk, M. S. (1981a). Description of finite measures on semifinite algebras. *Functional Anal. i Prilozhen*. **15**(3), 41–53. [in Russian]; (1981) *English Translation: Functional Analysis and Appli cation* **15**(3), 187–197. MR# 84h:46088.

Matvejchuk, M. S. (1981b). Finite signed measures on von Neumann algebras. In *Constructive theory of functions and functional analysis, Vol. III*, Kazan Cos. University Kazan. pp. 55–63. [in Russian], MR# 83i:46075.